QUANTUM SHANNON THEORY

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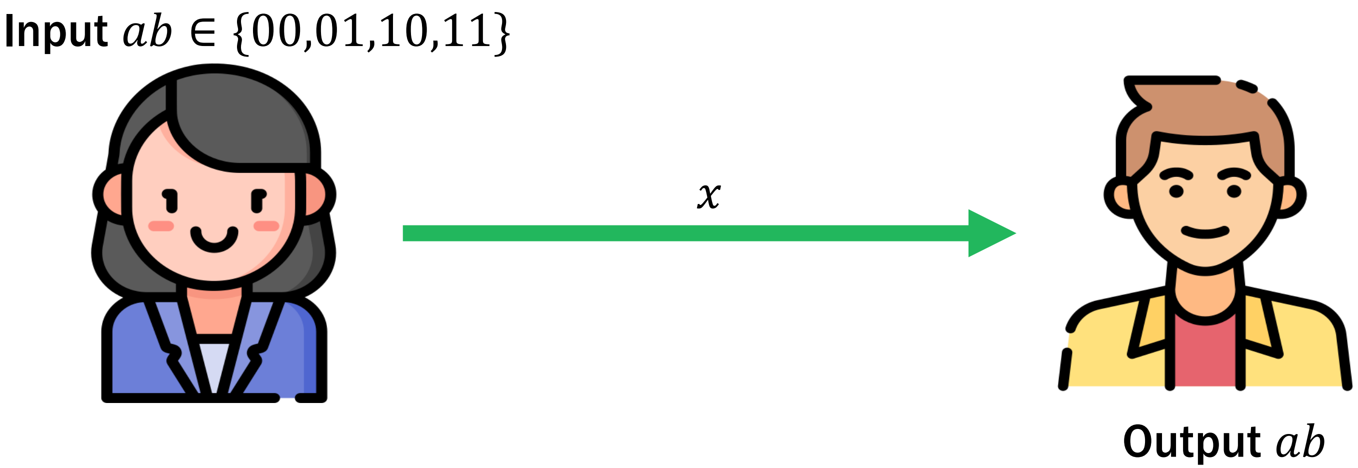
(ELL714: Basic Information Theory)

MOTIVATING EXAMPLE

Alice wants to convey **two** classical bits to Bob sending only **one** bit



Alice can convey both bits if she can send a qubit!

(given that they pre-share an entangled state)

AGENDA

Mathematical Formalism

Analogues of Shannon's Theorems:

- **Data Compression**
- **Channel Capacity**

STRANGE PROPERTIES OF QUANTUM INFORMATION

Quantum states cannot be **copied**

Quantum states cannot be **perfectly distinguished**

Quantum states can share **entanglement**

INFORMATION SOURCE

CLASSICAL

Modelled as a random variable X over a source alphabet Σ:

X	1	2	...	N
Pr(X)	p_1	p_2	...	p_N

QUANTUM

Modelled as a **density matrix** ρ over **quantum states:**

X	$\|\psi_1\rangle$	$\|\psi_2\rangle$	...	$\|\psi_N\rangle$
Pr(X)	p_1	p_2	...	p_N

Unit vectors in a finite dimensional vector space with inner product

$$\rho = \sum_x p_i|\psi_i\rangle\langle\psi_i|$$

EXAMPLE

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|+\rangle\langle +| = \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ 1) = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +| = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Properties of ρ:

- **Unit Trace**

$$\mathrm{tr}(\rho) = 1$$

- **Positive Semidefinite**

$$\langle \psi | \rho | \psi \rangle \geq 0$$

Ensemble: $\{p_i, \rho_i\}$

$$\rho = \sum_i p_i \rho_i$$

ENTANGLEMENT

A state of a composite system AB is separable if

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$$

Otherwise, it is said to be **entangled.**

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

"Spooky action at a distance"

Image source: Quantum Insider

FUNCTIONS OF OPERATORS

Spectral Decomposition Theorem

$$\rho = \sum_k \lambda_k |k\rangle\langle k|$$

λ_k: Eigenvalues (positive)

$|k\rangle$: Eigenvectors (orthonormal)

$$f(\rho) = \sum_k f(\lambda_k)|k\rangle\langle k|$$

$$\log(\rho) = \sum_k \log\lambda_k |k\rangle\langle k|$$

$$\rho\log(\rho) = \sum_k \lambda_k \log\lambda_k |k\rangle\langle k|$$

$$\boldsymbol{S(\rho) = -\mathrm{tr}(\rho\log\rho) = -\sum_k \lambda_k\log\lambda_k}$$

DATA COMPRESSION: SHANNON V/S SCHUMACHER

$\{X_i\}$: i.i.d. information source with Shannon entropy H(X)

R>H(X) $\leftrightarrow$ ∃ a reliable compression scheme of rate R

Alphabet	$\{1,2,3,\ldots\}$	$\{\|\phi_1\rangle, \|\phi_2\rangle, \|\phi_3\rangle, \ldots\}$
Information Source	X	$\rho = \sum_x p_x \|\phi_x\rangle\langle\phi_x\|$
Typical	**Sequence**	**Subspace**
Entropy	$H(X)$	$S(\rho)$

$\{|\psi_i\rangle\}$: i.i.d. **quantum** information source with **Von Neumann entropy S(ρ)**

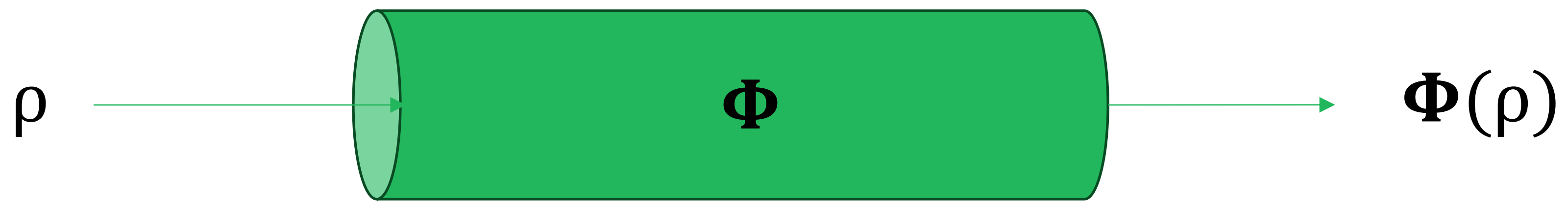
R>S(ρ) $\leftrightarrow$ ∃ a reliable compression scheme of rate R

QUANTUM CHANNELS

A channel which can transmit quantum and classical information.

Also known as a **quantum operation**

Modelled as a Completely Positive Trace Preserving Map:



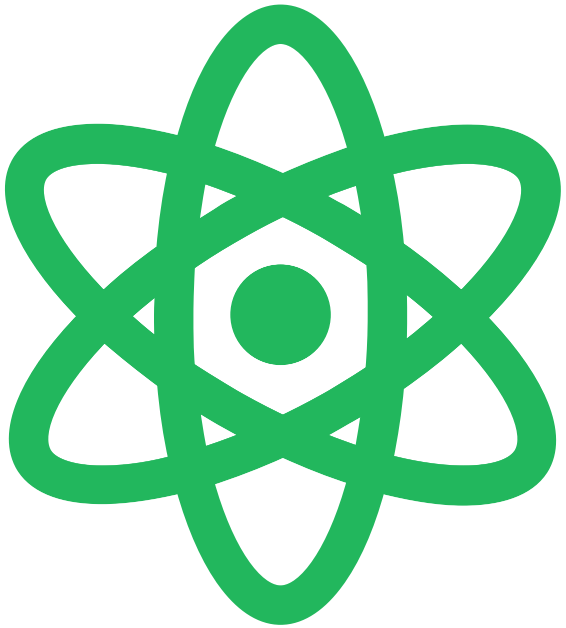
CLASSICAL INFORMATION VIA QUANTUM CHANNELS

Not completely solved.

If the sender can only produce product states:

Theorem [Holevo-Schumacher-Westmoreland (HSW)]

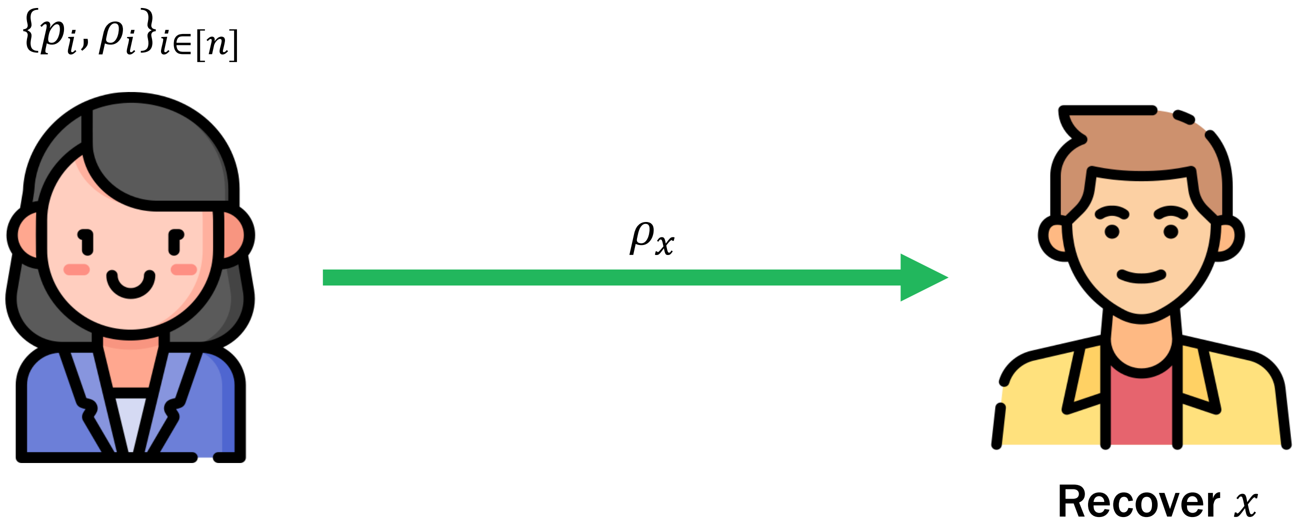
$$C^{(1)}(\Phi) = \max_{\{p_i,\rho_i\}} \left[S\left(\Phi\left(\sum_j p_j \rho_j \right) \right) - \sum_j p_j S(\Phi(\rho_j)) \right]$$



QUANTUM INFORMATION VIA QUANTUM CHANNELS?

THE HOLEVO BOUND

Alice wants to send a **classical message** to Bob by encoding it in a **quantum state.**



Whatever measurement Bob performs:

$$I(X:Y) \leq S(\rho) - \sum_i p_i S(\rho_i)$$

Corollary: n qubits cannot transmit more than n bits.


SO, QUANTUM TELEPORTATION—
THE NAME IS MISLEADING. IT'S A PARTICLE STATISTICS THING.
SO IT'S NOT LIKE STAR TREK? THAT'S BORING.
OKAY, I'M SICK OF THIS. EVERY TIME THERE'S A PAPER ON QUANTUM TELEPORTATION, YOU REPORTERS WRITE THE SAME DISAPPOINTED STORY.
BUT—
TALK TO SOMEONE ELSE. I'M GOING TO THE BAHAMAS.
CLICK
TELEPORTER
QUANTUM
REGULAR
VRMMM

THANK YOU